A closure model for intermittency in three dimensional incompressible turbulence

Piero Olla
The James Franck Institute and Ryerson Laboratories,
The University of Chicago, Chicago IL 60637

Abstract

A simplified Lagrangean closure for the Navier-Stokes equation is used to study the production of intermittency in the inertial range of three dimensional turbulence. This is done using localized wavepackets following the fluid rather than a standard Fourier basis. In this formulation, the equation for the energy transfer acquires a noise term coming from the fluctuations in the energy content of the different wavepackets. Assuming smallness of the intermittency correction to scaling allows the adoption of a quasi-gaussian approximation for the velocity field, provided a cutoff on small scales is imposed and a finite region of space is considered. In this approximations, the amplitude of the local energy transfer fluctuations, can be calculated self consistently in the model. Definite predictions on anomalous scaling are obtained in terms of the modified structure functions: $<< E(l,a) >_R^q>$, where $< E(l,a,\mathbf{r},t) >_R$ is the part of the turbulent energy coming from Fourier components in a band $(a-1)k$ around $k \sim l^{-1}$, spatially averaged over a volume of size $R \sim \frac{l}{a-1}$ around $\mathbf{r}$.

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I. Introduction

The statistics of large Reynolds numbers 3D (three dimensional) turbulence is characterized by scaling behaviors of the structure functions: $S(l,q) =< v_l^q >$, $\mathbf{v}_l(\mathbf{r},t) = \mathbf{v}(\mathbf{r}+\mathbf{l},t) - \mathbf{v}(\mathbf{r},t)$, which, in first approximation, appear to follow the Kolmogorov relation: $S(l,q) =\propto l^{\zeta_q^0}$, $\zeta_q^0 \simeq q/3$ [1]. However, both experiments [2] and numerical simulations [3] show the presence of corrections to Kolmogorov scaling, which become more and more pronounced for higher order moments. These corrections go in a direction of increasing nongaussianity of v_l as $l \to 0$: $S(l,q) \propto l^{\zeta_q}$, with $\delta\zeta_q = \zeta_q - \zeta_q^0$, satisfying the convexity condition: $\frac{\mathrm{d}^2\delta\zeta_q}{\mathrm{d}q^2} < 0$. Of course the relations above hold only for scales corresponding to the inertial range (and a part of the viscous range [4] if only ratios of moments are considered), so that for finite Reynolds numbers: $Re < \infty$, the generalized kurtosis $K(2q) = [S(l,q)/S(l,2)^q]_{l\to 0}$ is finite. However, a situation with scaling corrections in the form just described, persisting for $Re \to \infty$, would imply that $K(q > 2) \to \infty$ in this limit, corresponding to infinitely intermittent small scale velocity fluctuations.

The traditional approach to the issue of intermittency dates back to the remarks by Landau [5] on the presence of spatial fluctuations in the energy transfer between scales and its effect on the turbulent dynamics. From the Refined Similarity hypothesis of Kolmogorov [6], down the line to the Beta Model [7-8], the main ingredient is the assumption of a turbulent dynamics acting locally in scale, so that the energy density at a given point in space is essentially the product of coefficients, each with an independent fluctuating component, and each describing the transfer of energy between successive, contiguous scales. In this picture the source of intermittency lies in the inertial range and acts locally in scale; therefore, the scaling corrections is predicted to be independent of Re. This mechanism can be interpreted as a rule for the construction of a multifractal [9-10] in which the measure is given by: $d\mu_l(\mathbf{r},t) = l < (\partial_x v_x)^2 >_l (\mathbf{r},t)$, where $< >_l (\mathbf{r},t)$ indicates a spatial average taken at time t in a an interval of length l along the x-axis, centered around $\mathbf{r}$. In this way, the multifractal dimensions D_q, defined by the relation $< \mathrm{d}\mu_l^q > \sim l^{(q-1)D_q}$, are given by: $(q-1)(1-D_q) = q\zeta_2 - \zeta_{2q}$.

Recently, an alternative explanation for the presence of scaling corrections has been proposed, namely that what is observed is a finite size, i.e. a finite Reynolds number, effect [11-12]. This point of view is supported both by the smallness of the corrections and the strongly nonlocal character of finite size effects with respect to scale, as shown in numerical simulations of both Navier-Stokes dynamics [2] and reduced wavevector models [11]. In [12] a dynamical explanation for this effect was presented, based on closure arguments. The basic idea is that the strong intermittency of the viscous range, due to the interplay between dissipation and nonlinearity in the dynamics of small vortices [13], may be enough to generate, in the inertial range, scaling corrections of the appropriate size for all Reynolds numbers of practical interest. In this picture, the total intermittency, as parametrized by $K(q)$, would remain finite, while the scaling corrections would tend to zero in the infinite Reynolds number limit.

In this situation, it would be interesting to have some quantitative assessment of the magnitude of the finite size effects, relative to the amount of intermittency produced in the inertial range by energy transfer fluctuations. However, while [11] and [12] can afford some quantitative prediction on the size of the first effect, all models dealing with the second are heavily phenomenological and have no connection with real Navier-Stokes dynamics. An exception is the very recent paper by Yakhot [14], in which quantitative predictions have been obtained after approximating the turbulent energy dynamics with that of a passive scalar.

The purpose of the present paper is to obtain quantitative predictions on the Re-independent part of anomalous scaling (if present at all), following the route of evaluating the energy transfer fluctuations from statistical closure of the Navier-Stokes equation. The basic difficulty in using this technique is the real space nature of the quantities that have to be calculated: the moments $S(l,q)$ [15], while the main component of the turbulent dynamics, the energy transfer towards smaller scales, is better described in a Fourier basis. Although a real space closure of the Navier-Stokes equation was derived in [16], we found very difficult to extend the model to study energy fluctuations, especially when trying to separate contributions at different scales.

Here we prefer to follow the idea of Nakano [17] and, in a different context, of Eggers and Grossmann [18], of using a localized wavepacket representation of the Fourier-Weierstrass type rather than a global Fourier basis or a purely real space one. In this way, energy fluctuations on a scale R, instead of being

buried in the phase relationship between Fourier modes separated by $\Delta k \sim R^{-1}$, are described by the space dependence of the wavepackets. Now, if the wavepackets are centered around wavevectors lying in shells of radii $k_n = a^n k_0$, the spatial extension of a wavepacket at $k \sim l^{-1}$ will be $R(l) \sim l/(a-1)$. This means that a new object is being used in place of $S(l,q)$ to give a measure of intermittency: the generalized structure function $S(l,a,2q) =<< E(l,a) >^q_R>$, where $E(l,a,) = E(l,a;\mathbf{r},t)$ is the total energy of Fourier modes in the shell $k_n \sim l^{-1}$, and $R = R(l,a) = l/(a-1)$. More in general we shall consider the situation in which also the quantity $c_w = \frac{l}{(a-1)R}$ is treated as a free parameter, resulting in the definition of still another structure function $S(l,a,c_w,q)$ interpolating between the intermittency free limit $S(l,a,0,q)$ and the original case: $S(l,a,c_w^{\rm MAX},q) \equiv S(l,a,q)$.

The shell width $(a-1)k$ plays here a crucial role; this can be understood better by looking at how energy is transferred between shells as a rule for the construction of a multifractal. In this picture, a small value for $a-1$ corresponds to multipliers between one scale l and the next l/a being constant on domains $R(l) > l$ (in typical examples like the various "middle third" Cantor sets [10], one has $R(l) \simeq l$, i.e. the fluctuation scale and that of the geometrical structures is the same). In terms of turbulent dynamics, this has the following interpretation: $R(l)$ is the scale of the eddies which contribute the most to the straining of those of size l; the parameter $l^{-1}R(l) = (a-1)^{-1}$ gives therefore the degree of nonlocality of the nolinear interaction. In principle there will be fluctuations in the transfer of energy between eddies of size l also at scale smaller than $r(l)$ and this effect will result in correlations between phases of different wavepackets separated by $\Delta k > R^{-1}$. This effect contributes to the scaling of $S(l,q)$, but not to that of $S(l,a,q)$, and the last one is likely to underestimate the actual value of $\delta\zeta_q$.

Now, recent analysis performed by Domaradzki and Rogallo [19] on numerical simulations and statistical closures of the Quasi Normal Markovian type [20] has shown that there is indeed a separation of scales between straining flow and strained eddies, so that fluctuations in the transfer, occurring at scales similar to those of the modes exchanging energy, are not expected to be large, implying: $S(l,q) \sim S(l,a,q)$ even for a close to 1. More importantly, it is this separation of scales that allows to consider meaningful wavepackets, extending over several characteristic wavelengths, rather than having to deal with the usual eddy breaking picture, that is very attractive on grounds of simplicity, but does not allow to make any contact with the Navier-Stokes dynamics.

There is a second conceptual difficulty in using closures to study multifractal intermittency. This is the contradiction between an ansatz of quasi gaussianity and the "infinitely non gaussian" character of a multifractal with no cutoffs, as attested by the equation: $K(q > 2) = \infty$. A quasi gaussian hypothesis becomes meaningful however when studying limited regions of space and limited ranges of scales, which is possible if the degree of nonlocality of the nonlinear interactions is not too high and the intermittency correction $\delta\zeta_q$ is sufficiently small. In this way, although the distribution of values of, say $\partial_x v_x$, over different averaging volumes V_i is infinitely intermittent (if the total volume $V_{tot} = \cup V_i$ is infinite itself), the moments of $\partial_x v_x$ in each V_i will be close to gaussian [if the ultraviolet cutoff scale r is not too small: $(rV_i^{-1/3})^{\delta\zeta_q/q} - 1 \ll 1$]. Of course, in order for the statistical sample to be significant, it is necessary that the range of scales $r < l < V_i^{1/3}$ be large enough to accomodate all relevant interactions: $(a-1)V_i^{1/3}/r > 1$, and that enough wavepackets be present in the volume V_i: i.e. $(\frac{k}{(a-1)})^3 V_i \gg 1$. With a degree of nonlocality $(a-1)^{-1}$ of at most ten (see [19] and results in the next section), and $\frac{\delta\zeta_q}{q}$ in the range of a few percents [2], these conditions can be assumed to be satisfied, and all statistical averages, indicated by $< >$, will be understood here to be carried on in space, over a single large but finite volume V_i.

The closure that is going to be used in the next sections is of the Quasi-Lagrangean type [21], in which fluid structures at scale l are studied in a reference frame moving with a speed given by the average fluid velocity in a ball of radius l/λ, $\lambda < 1$; the free parameter λ is then adjusted to lead, in a mean field theory, to values of the Kolmogorov constant in agreement with experiments. This mean field theory is obtained using still a global Fourier basis. Notice that here l/λ is not the wavepacket size, and in this respect, the present approach differs from that of Nakano [17]. Of course our approximation is completely uncontrolled, in that sweep effects from scales between l and l/λ are still present, while part of the strain from scales larger than l/λ is lost. However, it is still an improvement upon using infrared cutoffs in the expression for the energy transfer, especially in view of the fact that this last operation would not preserve the nonlocal character of the nonlinear interaction and would lead to transfer profiles in disagreement with [19].

The fluctuating dynamics is obtained studying the energy equation for the wavepackets [which, after angular integration, becomes the energy equation for the shells $k_n < k < k_{n+1}$]. This is obtained by substituting the statistical averages which would lead to a mean field closure, with partial spatial averages over wavepackets volumes. This leads to an energy equation in the same form (in Fourier space) as the original mean field one, plus a noise term that is essentially: $< v^{(0)}(v^{(0)}\nabla)v^{(0)} >_R (\mathbf{r},t)$. Here $v^{(0)}$ is the gaussian random field, which gives the lowest order approximation for the fluid velocity. The end result is an equation for the energy content of shells at a certain position in space, which is in a form very close to the stochastic chains studied by Eggers in [22-23].

This paper is organized as follows. The closure technique is going to be described in section II and equations for the energy balance in laboratory as well in Lagrangean frame are derived in the mean field approximation. In section III, this closure is applied to the analysis of the energy transfer fluctuations. In section IV, an energy balance equation in terms of shells is derived and its solution is used to obtain the intermittency corrections to scaling. Section V is devoted to discussion of the results and to concluding remarks. Most of the calculation details have been confined to the appendices.

II. Mean energy transfer in Lagrangean reference frame

A. Closure outline

In the inertial range of fully developed 3D turbulence, the dynamics obeys the Euler equation:

$$\partial_t \mathbf{v}(r,t) + (\mathbf{v}\cdot\nabla)\mathbf{v}(\mathbf{r},t) + \nabla P(\mathbf{r},t) = 0, \tag{1}$$

while the presence of dissipation is accounted for by the energy flux towards small scales; the pressure P is calculated through the incompressibility condition $\nabla\cdot\mathbf{v} = 0$. In a Lagrangean reference frame, Eqn. (1) can be rewritten in the equivalent form:

$$D_t\mathbf{v}(\mathbf{z}_t+\mathbf{r},t) = ((\mathbf{v}(\mathbf{z}_t,t) - \mathbf{v}(\mathbf{z}_t+\mathbf{r},t))\cdot\nabla)\mathbf{v}(\mathbf{z}_t+\mathbf{r},t) - \nabla P(\mathbf{z}_t+\mathbf{r},t) = 0, \tag{2}$$

where $D_t = \partial_t + \mathbf{v}(\mathbf{z}_t,t)\cdot\nabla$ is the material derivative along the trajectory $\mathbf{z}_t = \mathbf{z}_t(\mathbf{r}_0,t_0)$ of a Lagrangean tracer that at time t_0 was at $\mathbf{r}_0$.

The first assumption of the model is that the velocity field can be taken to lowest order to obey gaussian statistics, with correlation given, in a Lagrangean frame (with corrections to the exponential decay in t for small t), by the expression:

$$\begin{aligned} C_k^{\alpha\gamma}(t) &\simeq \int \mathrm{d}^3 r\, e^{i\mathbf{k}\cdot\mathbf{r}} C^{\alpha\gamma}(\mathbf{r},t;0,0) \\ &= 2\pi^2 C_{Kol}\bar{\epsilon}^{2/3}P^{\alpha\gamma}(\mathbf{k})k^{-11/3}\exp(-\eta_k t). \end{aligned} \tag{3}$$

$$C_{\alpha\gamma}(\mathbf{r},t;\mathbf{r}_1,t_1) \equiv < v_\alpha^{(0)}(\mathbf{z}_t+\mathbf{r},t)v_\gamma^{(0)}(\mathbf{z}_{t_1}+\mathbf{r}_1,t_1) >,$$

where $t>0$, C_{Kol} is the Kolmogorov constant, $\bar{\epsilon}$ the mean energy dissipation, $P^{\alpha\gamma}(\mathbf{k}) = \delta^{\alpha\gamma} - \frac{k^\alpha k^\gamma}{k^2}$ the transverse projector and $\eta_k = \rho\bar{\epsilon}^{1/3}k^{2/3}$ the eddy turnover frequency at scale k^{-1}, with ρ a dimensionless constant. The statistical average $< >$ taken in Eqn. (3) can be understood as an average over the initial position $\mathbf{r}_0$, which is taken in a large but finite volume, as discussed in the introduction.

The first nongaussian correction $\mathbf{v}^{(1)}$ is obtained from (2); this is integrated, keeping second order terms, in the Direct Interaction Approximation (DIA) [24-26], rather than following [16]. The result contains then a green function: $G(t,\mathbf{r}|t_1,\mathbf{r}_1)$, which gives the effect of a source, which at time t_1 was in $\mathbf{z}_{t_1}+\mathbf{r}_1$, on a point $\mathbf{z}_t+\mathbf{r}$ at time t:

$$\begin{aligned} v_\alpha^{(1)}(\mathbf{r}_0+\mathbf{r},0) = &\int_{-\infty}^0 \mathrm{d}t \int \mathrm{d}^3 s\, G_\alpha^\rho(0,\mathbf{r}|t,\mathbf{r}+\mathbf{s}) \\ &\times [v_\sigma^{(0)}(\mathbf{z}_t+\mathbf{r}+\mathbf{s},t) - v_\sigma^{(0)}(\mathbf{z}_t,t)]\partial^\sigma v_\rho^{(0)}(\mathbf{z}_t+\mathbf{r}+\mathbf{s},t). \end{aligned} \tag{4}$$

The time decay of the (retarded) green function introduced above is assumed to be the same as that of the correlation function [26] and the following approximation is adopted:

$$G(t,\mathbf{r}|t_1,\mathbf{r}_1)\simeq G(t,\mathbf{r}-\mathbf{r}_1|t_1,0)\equiv G(\mathbf{r}-\mathbf{r}_1,t-t_1), \tag{5}$$

which means that the initial and final position in G are shifted together until the first lies on the Lagrangean trajectory $\mathbf{z}_t(\mathbf{r}_0,t_0)$. Eqns. (5) allows a great simplification in taking Fourier transforms, in that all correlations become diagonal: $<v_{\mathbf{p}}v_{\mathbf{q}}>\propto\delta(\mathbf{p}-\mathbf{q})$. In particular, it becomes possible to write time correlations in the form: $C_k^{\alpha\gamma}(t)=G^\alpha_{\beta,k}(t)C_k^{\beta\gamma}(0)$, where:

$$G_k^{\alpha\gamma}(t)=\int \mathrm{d}^3re^{i\mathbf{k}\cdot\mathbf{r}}G^{\alpha\gamma}(\mathbf{r},t)=P^{\alpha\gamma}(\mathbf{k})\exp(-\eta_k|t|). \tag{6}$$

The meaning of this approximation is that the divergence of Lagrangean trajectories is disregarded. In particular, the statistical average over the initial position $\mathbf{r}_0$ coincides identically with spatial average at the given time. We have also the result that, in this approximation, the advanced green function coincides with the "transpose", with respect to the space slots, of the retarded one; in the following sense: for $t_1>t_2$,

$$<v_\alpha^{(0)}(\mathbf{z}_{t_1}+\mathbf{r}_1,t_1)v_\gamma^{(0)}(\mathbf{z}_{t_2}+\mathbf{r}_2,t_2)>=\int \mathrm{d}^3sG^\sigma_\gamma(t_1,\mathbf{r}_2+\mathbf{s}|t_2,\mathbf{r}_2)C_{\alpha\sigma}(\mathbf{r}_2+\mathbf{s}-\mathbf{r}_1,0)$$

$$=\int \mathrm{d}^3sG^\sigma_\alpha(t_1,\mathbf{r}_1|t_2,\mathbf{r}_1+\mathbf{s})C_{\sigma\gamma}(\mathbf{r}_1+\mathbf{s}-\mathbf{r}_2,0). \tag{7}$$

Further discussion of these points is contained in Appendix A.

The final assumption is necessary in order to separate sweeping from straining scales, and is that, inside averages, one has to carry on the following substitution:

$$<...[\mathbf{v}(\mathbf{z}+\mathbf{r})-\mathbf{v}(\mathbf{z})]\cdot\nabla...>_k\rightarrow<...[\mathbf{v}(\mathbf{z}+\mathbf{r})-\hat{w}(\lambda,k)\mathbf{v}(\mathbf{z})]\cdot\nabla...>_k, \tag{8}$$

where $\hat{w}(\lambda,k)$ is a smoothing operator acting on $\mathbf{v}(\mathbf{z})$ by filtering out Fourier modes above λk. The hypothesis here is that, although all velocity components are integrated along a single Lagrangean trajectory $\mathbf{z}_t(r_0,t_0)$, when these components are large scale ones, small scale details of $\mathbf{z}_t$ will not contribute in averages.

B. Mean field analysis

In order to study the fluctuation dynamics in the energy transfer T, we will have to work in a Lagrangean frame. However, in order to fix the free parameter λ introduced above, it is necessary to match theoretical predictions with experimental data of the Kolmogorov constant that are taken in a fixed laboratory frame. For this reason, the first step in the analysis is the derivation of closure equations for eulerian correlations.

The calculations to obtain the energy equation in the laboratory frame are standard [26]: by multiplying Eqn. (1) by $\mathbf{v}(\mathbf{r}_0,0)$ and taking the average, the pressure term drops off because of incompressibility, and one is left with:

$$\partial_tC^{\alpha\gamma}(\mathbf{r},t)|_{t=0}=-<v^\alpha(0,0)v^\beta(\mathbf{r},0)\partial_\beta v^\gamma(\mathbf{r},0)>, \tag{9}$$

which of course is equal to zero at steady state. Actually, this is the equation for the Eulerian 2-time correlation at zero time separation; the energy equation is obtained from (9) by multiplying its RHS (right hand side) by 2. Expansion to first order in $v^{(1)}$ and use of Eqns. (3-4) and (7), leads to the same expression for the energy equation as in DIA (both Eulerian and Lagrangean, [24-26]), and in "Quasi Normal-Markovian" closures [20]:

$$\left.\frac{\partial C_k(t)}{\partial t}\right|_{t=0}=\Big(\frac{\pi}{k}\Big)^2T(k)=\frac{1}{4\pi^2}\int_\Delta \mathrm{d}p\mathrm{d}q\,kpq\,\theta_{kpq}b_{kpq}C_q(C_p-C_k), \tag{10}$$

where: $C_k^{\alpha\gamma}(t)=P^{\alpha\gamma}(\mathbf{k})C_k(t)$; $C_k\equiv C_k(0)$; Δ is the domain defined by the triangle inequalities: $p>0$ and $|k-p|<q<k+p$; $\theta_{kpq}=(\eta_k+\eta_p+\eta_q)^{-1}$ is the relaxation time; $b_{kpq}=(p/k)(xy+z^3)$ is the geometric factor, in which x,y and z are cosines of the angles opposite respectively to k,p and q in a triangle with sides kpq. The terms associated with integrating along $\mathbf{z}$ are uniform in space, they do not couple with the others

and are shown in Appendix A not to contribute to the final result: in this model, the choice of integrating back in time along a Lagrangean path is felt only in the eddy turnover time θ_{kpq}.

The green function G is a Lagrangean object and is obtained by multiplying this time Eqn. (2) by $\mathbf{v}(\mathbf{r}_0, 0)$ and then taking averages. The resulting triplet term is in the form: $< v^\alpha(\mathbf{r}_0, 0)[v^\beta(\mathbf{z}_t + \mathbf{r}, t) - v^\beta(\mathbf{z}_t, t)]\partial_\beta v^\gamma(\mathbf{z}_t + \mathbf{r}, t) >$; now, $v^\beta(\mathbf{z}_t, t)$, that is the term coming from the shift to a Lagrangean frame, contributes to the final expression and is responsible for the cancellation of the sweep terms. Substituting Eqns. (3-4) and (7) in the new triplet, and expanding again to first order in $v^{(1)}$, we obtain at steady state the following equation:

$$\frac{DC_k(t)}{Dt} = \frac{1}{4\pi^2}\int_0^t \mathrm{d}\tau \int_\Delta \mathrm{d}p\mathrm{d}q[b_{kpq}G_p(t-\tau)C_k(\tau)C_q(t-\tau)$$

$$+w_p(\lambda,k)b^{(1)}_{kpq}G_p(t-\tau)C_k(t-\tau)C_q(\tau) - w_q(\lambda,k)b^{(2)}_{kpq}G_k(t-\tau)C_p(\tau)C_q(t-\tau)] \tag{11}$$

where: $G^{\alpha\gamma}_k = P^{\alpha\gamma}G_k$, $b^{(1)}_{kpq} = (p/2k)(xy(1-2z^2) - y^2 z)$ and $b^{(2)}_{kpq} = (p/2k)(xy + z(1 + z^2 - y^2))$ are new geometric terms, and $w_q(\lambda, k)$ gives the effect in k-space of the cutoff operator $\hat{w}(\lambda, k)$ The term in Eqn. (11), which cancels the divergence of the integral for $q \to 0$, is the one in $b^{(2)}_{kpq}$. Integrating from $t = 0$ to $t = \infty$ and using Eqns. (3) and (6), we get the result:

$$\frac{\rho^2}{C_{Kol}} = \frac{1}{2}\int_\Delta \mathrm{d}p\mathrm{d}q\, pq^{-8/3}\Big[\frac{b_{kpq}}{p^{2/3}+q^{2/3}} + \frac{w_p b^{(1)}_{kpq}}{(1+p^{2/3})q^{2/3}} - \frac{w_q b^{(2)}_{kpq}}{(1+q^{2/3})p^3}\Big], \tag{12}$$

which gives a first equation connecting C_{Kol} and ρ with λ. A second equation connecting C_{Kol} and ρ, given an energy balance equation in the form of (10), was derived by Kraichnan [27]:

$$\rho/C^2_{Kol} \simeq 0.19. \tag{13}$$

The constant C_{Kol} considered in this section is a well defined quantity, provided the average volume V_i and the range of scales k are not too large; in this sense we have locally: $C_k(V_i) \propto C_{Kol}\bar{\epsilon}_i^{2/3}k^{-11/3}$, even though for $V_i \to \infty$ anomalous corrections become important, and C_{Kol} ceases to have a clear meaning. Imposing that the Kolmogorov constant matches the experimentally observed value $C_{Kol} \approx 1.5$ (we are assuming here that that the finite size effects and intermittency corrections which affect the experimental C_{Kol} are indeed small), and using a gaussian profile for the cutoff w: $w_p(\lambda, k) = \exp(-(p/\lambda k)^2)$, Eqns. (12) and (13) set: $\lambda = 0.9$. In the following we will fix therefore:

$$w_p(\lambda, k) \to w_p(k) = \exp(-1.23(p/k)^2). \tag{14}$$

The same steps leading from Eqns. (1) to (10) can be repeated starting from Eqn. (2). The result is the following equation for velocity correlations in a Lagrangean frame:

$$\frac{DC_k(t)}{Dt}\Big|_{t=0} = \Big(\frac{\pi}{k}\Big)^2 T(\lambda', k) =$$

$$\frac{1}{4\pi}\int_\Delta \mathrm{d}p\mathrm{d}q\, kpq\, \theta_{kpq}[b_{kpq} + w_q(\lambda', k)b^{(3)}_{kpq}]C_q(C_p - C_k), \tag{15}$$

where: $b^{(3)}_{kpq} = (p/k)(1 + xy + z(z^2 - \frac{x^2+y^2}{2}))$ [28]. Again, the corresponding energy equation is obtained by multiplying the RHS of (15) by 2 and understanding the t on the LHS (left hand side) not as a time separation in a 2-time correlation but as the t-dependence in a non stationary 1-time correlation. Notice that the parameter λ' entering Eqn. (15) does not have to coincide with $\lambda = 0.9$, which is fixed and separates between sweeping and straining scales. This allows to calculate the Kolmogorov constant and the parameter ρ in different reference frames. The result is shown in Fig. 1. and suggests that, although differences are expected between quantities measured in laboratory and Lagrangean frame (due to correlation among reference frame velocity and quantity to be averaged), their orders of magnitude should be the same. This is

particularly important because the intermittency estimates that are going to be derived in the next sections will have to be calculated in a Lagrangean, not a laboratory frame.

The structure of the energy transfer is studied, following Domaradzki and Rogallo, [19] by decomposing $T(\lambda,k)$ in its contribution from different scales:

$$T(\lambda,k)=\int \mathrm{d}p\,T(\lambda,k,p). \tag{16}$$

It appears that the present closure is able to maintain the features of large scale straining observed in [19] also in a Lagrangean frame, as it should be expected. It is clear instead, from Fig. 2., that simpler closures based on a Navier-Stokes nonlinearity in k-space, amputated of the large scale convection contributions:

$$(k_\alpha P_{\beta\gamma}(\mathbf{k})+k_\gamma P_{\beta\alpha}(\mathbf{k}))v^\alpha_{\mathbf{p}}v^\gamma_{\mathbf{q}}\to(k_\alpha(1-w_q)P_{\beta\gamma}(\mathbf{k})+k_\gamma(1-w_p)P_{\beta\alpha}(\mathbf{k})))v^\alpha_{\mathbf{p}}v^\gamma_{\mathbf{q}}, \tag{17}$$

would lead to transfer profiles almost without any exchange of energy between nearby scales.

III. Energy transfer fluctuations

Although the analysis in section II disregarded fluctuations, all averages were implicitly dependent on the position (through $\bar\epsilon$), at the scale of the volumes V_i. In this section, we calculate the same averages over balls of radius R and the variations of the result from ball to ball are used to study the fluctuations of the energy transfer. If the ratio $RV_i^{-1/3}$ is not too large, these fluctuations are going to be small and the analysis can be carried on in perturbation theory.

In principle, it would be nice to carry on the calculations in the laboratory frame, where experimental data are available. However, the statistics of the velocity field is ill defined there, due to the effect of sweep. The balls are then imagined to move rigidly along Lagrangean trajectories passing through their centers at time t_0, while, thanks to the simplifying assumptions of Eqns. (6) and (7), the average over the initial position is substituted by a spatial average inside the balls. As discussed in the introduction, we associate to each scale l, a certain radius $R(l)\sim l/(a-1)$; this automatically induces a basis of wavepackets of width $\Delta k\sim R^{-1}$. In particular, we obtain a partition of Fourier space in shells $\mathtt{S}_n$ of radii $k_n=a^nk_0$, each containing $\sim(k_n/\Delta k)^2$ wavepackets, associated with the different orientations of the wavevector $\mathbf{k}$ [17]. Clearly, a further degeneracy in the wavepackets is produced by their different location in real space.

For R/l large, the main interactions occur among overlapping balls and are associated with local transfer of energy towards small scales. We consider therefore a sequence of nested balls and study the transfer of energy among them. Notice that a derivation of deterministic equations for a shell model of the type considered here, would require some dynamical analog of the statistical assumptions of Eqns. (5-7); at this point it is more natural to follow the route of statistical closure to the end.

Let us indicate with $<\ >_m$ the spatial average over $\mathtt{B}_m$: the ball of radius R_m associated with the m-th shell; once we have fixed the origin of the axis of the moving frame on $\mathbf{z}_t$, we can write $<\ >_m$ in terms of a kernel $W(r,m)$:

$$<\Psi>_m=\int \mathrm{d}^3r\,W(r,m)\Psi(\mathbf{r})=\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}W_k(m)\Psi_{\mathbf{k}},$$

In this way, the energy density in $\mathtt{B}_m$ reads:

$$E(m)=\int \mathrm{d}k\,E_k(m);\qquad E_k(m)=2\pi k^2\int\frac{\mathrm{d}^3p}{(2\pi)^3}W_p(m)\mathbf{v}_{\frac{\mathbf{p}}{2}+\mathbf{k}}\cdot\mathbf{v}_{\frac{\mathbf{p}}{2}-\mathbf{k}}, \tag{18}$$

To fix our ideas we shall consider gaussian wavepackets:

$$W_k(m)=\exp(-(k/\Delta k_m)^2),\qquad \Delta k_m=c_w(a-1)k_m, \tag{19}$$

with c_w relating the wavepacket thickness Δk_m and the shell spacing $k_{m+1}-k_m=(a-1)k_m$. Notice that c_w can be treated as a free parameter, in that, for a fixed choice of shell (i.e. for a given value of a), we can

still consider arbitrarily thin wavepackets, or in the limit, even global Fourier modes. Of course, it is the thickest wavepacket for a given a (that is the maximum value of c_w), which will be able to catch most of the fluctuation dynamics.

If $k_n R_m$ is large, $W_p(m) \propto R_m^{-3}\delta(\mathbf{p})$ and we can approximate:

$$E_k(m) \simeq (1+\phi_n(m))E_k^{(0)} \equiv (1+\phi_n(m))C_{Kol}\bar{\epsilon}^{2/3}k^{-5/3}, \tag{20}$$

with $E_k^{(0)} = C_k k^2/2\pi^2$ the spectral energy density in V_i, $n = n(\mathbf{k})$ the shell of the wavevector $\mathbf{k}$, and $\phi_n(m) = \phi_n(m,t)$ fluctuating and small.

The energy density $E(m)$ can be expressed also as a sum of contributions from different shells:

$$E(m) = \sum_n E_n(m); \qquad E_n(m) = \int_n \mathrm{d}k E_k(m). \tag{21}$$

The term $E_n(m) \sim < E(k_n^{-1}, a, \mathbf{z}_t, t) >_m$ can be seen as the istantaneous total energy of wavepackets in $\mathtt{S}_n$, lying in the volume $\mathtt{B}_m$ at $\mathbf{z}_t$; however, if $\mathtt{B}_m$ becomes smaller than the wavepackets, i.e. (for c_w maximal) $n > m$, the only average taking place will be over wavevector orientations and will be independent of m; hence we get: $\phi_n(m<n) = \phi_n(n)$.

With these definitions, the energy equation for the shells reads:

$$D_t E_n(m) = T_n(m) + f_n(m). \tag{22}$$

In the equation above, the energy transfer into $\mathtt{S}_n$, averaged over $\mathtt{B}_m$, has two components. The first:

$$T_n(m) = \int_n \mathrm{d}k T'(k), \tag{23}$$

is a relaxation term giving the response of the system to fluctuations in the energy content of the various shells. For small ϕ, the integrand $T'(k)$ coincides with the transfer term $T(\lambda, k)$ of Eqn. (15), with the substitution: $C_k \to (1+\phi_n(m))C_k$. For $c_w \to c_w^{MAX}$, the dk integral in Eqn. (23) receives discreteness corrections from the shell and the wavepacket thickness being comparable; notice however that the $\int_\Delta \mathrm{d}p\mathrm{d}q$ integral contained in $T'(k)$ [see Eqns. (10) and (15)] remains unaffected, when $a-1$ is small.

The second term contains the fluctuations of the transfer, around the average. To lowest order in the expansion around gaussian statistics it has itself two components. The first comes from the pressure: $< \mathbf{v}\cdot\nabla P >_m$ and results in a surface integral over the boundary of $\mathtt{B}_m$. The second: $< \mathbf{v}^{(0)}\cdot(\mathbf{v}^{(0)}\cdot\nabla)\mathbf{v}^{(0)} >_m$ did not contribute in the mean field analysis of the previous section, because of the gaussianity of $\mathbf{v}^{(0)}$; it plays a role here however, by acting as a source of fluctuations in Eqn. (22). Now, we are considering a situation in which kR is large; hence, the pressure contribution can be neglected and we are left with:

$$f_n(m) = -< \hat{h}(n)[\mathbf{v}^{(0)}(\mathbf{z}_t,t)\cdot((\mathbf{v}^{(0)}(\mathbf{z}_t+\mathbf{r},t) - \mathbf{v}^{(0)}(\mathbf{z}_t,t))\cdot\nabla)\mathbf{v}^{(0)}(\mathbf{z}_t+\mathbf{r},t)] >_m, \tag{24}$$

where $\hat{h}(n)$ is a band pass filter for the modes in $\mathtt{S}_n$: $\hat{h}(n)f(\mathbf{r}) \equiv \int \mathrm{d}^3r' h(n,r')f(\mathbf{r}+\mathbf{r}') = \int \frac{\mathrm{d}^3k}{(2\pi)^3}h_k(n)f_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}$, with $h_k(n) = H(k-k_n) - H(k-k_{n+1})$; $H(x)$ is the Heaviside step function.

Treating $f_n(m)$ as an external (nongaussian) noise, Eqn. (22) becomes a stochastic differential equation for the $\phi_n(m)$'s. Following Eggers [22-23], we express $f_n(m)$ as the difference between the fluctuations in the energy flux across k_n and k_{n+1}:

$$f_n(m) = g_n(m) - g_{n+1}(m), \tag{25}$$

where g_n and g_{n+1} are associated one with each of the step functions entering the definition of $h_k(n)$.

A. Fluctuation source

The statistics of $g_n(m)$ can be calculate explicitly from Eqns. (3) (24) and (25). Here we shall content ourselves with the analysis of the 2-point correlations. Already, this calculation requires the evaluation of some fifteen contractions of the product:

$$v^\alpha(\mathbf{z}_1)[v^\beta(\mathbf{z}_1+\mathbf{r}_1) - v^\beta(\mathbf{z}_1)]\partial_\beta v^\alpha(\mathbf{z}_1+\mathbf{r}_1)v^\gamma(\mathbf{z}_2)[v^\sigma(\mathbf{z}_2+\mathbf{r}_2) - v^\sigma(\mathbf{z}_2)]\partial_\sigma v^\gamma(\mathbf{z}_2+\mathbf{r}_2); \tag{26}$$

these contractions are listed in table B1. in Appendix B. Of these only six contribute; one example is contraction (B1.9):

$$
\begin{aligned}
&< g_{n_1}(m_1,t_1) g_{n_2}(m_2,t_2) >^{(9)}= \\
&\int \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 H(r_1,n_1) H(r_2,n_2) \int \mathrm{d}^3 z_1 \mathrm{d}^3 z_2 W(z_1,m_1) W(z_2,m_2) C^{\alpha\gamma}(\mathbf{z}_1-\mathbf{z}_2,t) \\
&\times \partial_\sigma [C^{\beta\gamma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t) - \hat{w} C^{\beta\gamma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)] \\
&\times \partial_\beta [C^{\alpha\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t) - \hat{w} C^{\alpha\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)];
\end{aligned} \tag{27}
$$

notice the cutoff operators $\hat{w}$ signaling a term coming from working in a Lagrangean reference frame. In terms of Fourier components, we get:

$$
\begin{aligned}
&< g_{n_1}(m_1,t_1) g_{n_2}(m_2,t_2) >^{(9)}= \\
&\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} C_k C_p C_q \exp(-(\eta_k+\eta_p+\eta_q)|t|) \\
&\times W_{\mathbf{p}+\mathbf{q}-\mathbf{k}}(m_1) W_{\mathbf{p}+\mathbf{q}-\mathbf{k}}(m_2)\, pkyz(y+xz) \\
&\times [(2-w_p(k)) H(k-k_{n_1}) - w_p(k) H(q-k_{n_1})] \\
&\times [(2-w_q(k)) H(k-k_{n_2}) - w_q(k) H(p-k_{n_2})].
\end{aligned} \tag{28}
$$

(The wavevectors entering the two last lines of Eqn. (28) can be tracked back to Eqn. (26) and (B1.9): the p and k in $w_p(k)$ come from v_p^β and v_k^α, while the k and q in $H(k-k_{n_1})$ and $H(q-k_{n_1})$ come from v_k^α and v_q^α). It has already been mentioned that, for large kR, the averaging kernel W is proportional in k-space to a Dirac delta. Also the product $W(m_1)W(m_2)$ is a Dirac delta:

$$
W_{\mathbf{k}}(m_1) W_{\mathbf{k}}(m_2) \simeq (c_w(a-1))^3 \Big(\frac{\pi\, k_{m_1}^2 k_{m_2}^2}{k_{m_1}^2 + k_{m_2}^2} \Big)^{3/2} \delta(\mathbf{k}), \tag{29}
$$

where use has been made of Eqn. (19). Eqn. (29) allows simplification of Eqn. (28) by means of the bipolar integral formula [26]: $\int \mathrm{d}^3 p \mathrm{d}^3 q \delta(\mathbf{p}+\mathbf{q}-\mathbf{k}) = \frac{2\pi pq}{k} \int_\Delta \mathrm{d}p \mathrm{d}q$. Repeating the calculations leading to (28) with all the other contractions we obtain the result:

$$
\begin{aligned}
&< g_{n_1}(m,t_1) g_{n_2}(m,t_2) >= \frac{\pi^{\frac{1}{2}} \bar{\epsilon}^2}{2^{\frac{9}{2}}} [C_{Kol}\, c_w\, (a-1) k_m]^3 \\
&\times \int_0^\infty \mathrm{d}k \int_\Delta \mathrm{d}p \mathrm{d}q\, (kpq)^{-8/3} \exp(-(\eta_k+\eta_p+\eta_q)|t|) \\
&\times [H_1 B_1(kpq) + H_2 B_2(kpq) + H_3 B_3(kpq)],
\end{aligned} \tag{30}
$$

The terms B_i $i=1,2,3$ are geometrical factors similar to the b_{kpq} terms entering the expression for the transfer function. The factors $H_i = H_i(n_{1,2}; k,p,q)$ are expressed in terms of step functions and restrict the integrals to the appropriate domains for the calculation of Lagrangean frame energy fluxes; they have the same origin as the two last lines of Eqn. (28). The exact form of both the B_i and H_i functions is given in Eqns. (B4-7) of Appendix B.

For $t=0$, Eqn. (30) can be reduced to double integrals using the standard change of variables [26]: $k = k_n/u$, $p = k_n v/u$, $q = k_n w/u$ and exploiting the similarity of the integrand with respect to u. Similarly, the following expression for the noise correlation time:

$$
\bar{\eta}_n(m)^{-1} = < f_n(m)^2 >^{-1} \int_0^\infty \mathrm{d}t < f_n(m,t) f_n(m,0) >,
$$

can be reduced, after explicit calculation of the time integral, to a double integral. Numerical evaluation gives then the results:

$$
< g_n(m)^2 > \simeq 0.15\, (C_{Kol}\, c_w\, (a-1))^3 a^{l(n,m)} \bar{\epsilon}^2; \tag{31}
$$

and:

$$\bar{\eta}_n(m) = \bar{\rho}\bar{\epsilon}^{1/3}k_n^{2/3} \qquad \bar{\rho} \simeq 0.6, \tag{32}$$

where $l(n,m) = \min(0, m-n)$ for c_w maximal. The decay of the correlation with respect to scale is shown in Fig. 3. Notice that $\bar{\rho} \simeq 2\rho_{\lambda=0.9}$ (see Fig. 1.), which is what one would get identifying naively: $< g(0)g(t) > \propto < v(0)v(t) >^2$. Similarly, the decay of $< g_n g_{n'} >$ with respect to $(n-n')\ln a$ is approximately the same as that of the auto-correlation for $|T(k,p)|$: $A(q) = \int \mathrm{d}p|T(k,p)T(k+q,p+q)|$, which is due to the fact that the energy flux at scale k is roughly equal to $\int \mathrm{d}p|T(k,p)|$.

B. Relaxation term

The shell energy equation (22) is in the form of a nonlinear stochastic differential equation. Since ϕ is small, we can linearize $T_n(m)$ (including the frequencies $\eta_k \sim k^{\frac{3}{2}}E_k^{\frac{1}{2}}$ entering the term θ_{kpq}), with a result that is similar to the stochastic model of Eggers [22]. One difference is the presence here of energy transfer between non adjacent shells. To analyze this issue, we write the transfer $T_n(m)$ as a sum of contributions in which the wavevectors k p and q are respectively in $\mathtt{S}_n$, $\mathtt{S}_r$ and $\mathtt{S}_s$:

$$T_n(m) = \frac{C_{Kol}^2\bar{\epsilon}}{\rho}\sum_{r,s} T_{nrs}(m); \qquad T_{nrs} = c_{nrs}\phi_r + c_{nsr}\phi_s - d_{nrs}\phi_n, \tag{33}$$

where, from Eqns. (15) and (20):

$$c_{nrs} = \int_n \mathrm{d}k \int_{\Delta_{rs}} \mathrm{d}p\mathrm{d}q\, \tilde{\theta}_{kpq}\Big[\tilde{a}_{kpq}k^2(pq)^{-5/3} - \Big[\frac{2}{3}p^{2/3}\tilde{\theta}_{kpq}\Big(\tilde{a}_{kpq}k^2(pq)^{-5/3}$$

$$-\tilde{b}_{kpq}p^2(kq)^{-5/3} - \tilde{b}_{kqp}q^2(kp)^{-5/3}\Big) + \tilde{b}_{kpq}q^2(kp)^{-5/3}\Big]\Big], \tag{34}$$

and:

$$d_{nrs} = \int_n \mathrm{d}k \int_{\Delta_{rs}} \mathrm{d}p\mathrm{d}q\, \tilde{\theta}_{kpq}\Big[\frac{2}{3}k^{2/3}\tilde{\theta}_{kpq}\Big(\tilde{a}_{kpq}k^2(pq)^{-5/3}$$

$$-\tilde{b}_{kpq}p^2(kq)^{-5/3} - \tilde{b}_{kqp}q^2(kp)^{-5/3}\Big) + \tilde{b}_{kpq}p^2(kq)^{-5/3} + b_{kqp}q^2(kp)^{-5/3}\Big]. \tag{35}$$

The quantities appearing on the RHS of Eqns. (33-34) are defined as follows: Δ_{rs} is the restriction of Δ to $p \in \mathtt{S}_r$; $q \in \mathtt{S}_s$; $\tilde{b}_{kpq} = b_{kpq} + w_q(k)\, b^{(3)}_{kpq}$; $\tilde{a}_{kpq} = \tilde{b}_{kpq} + \tilde{b}_{kqp}$ and $\tilde{\theta}_{kpq} = (k^{2/3} + p^{2/3} + q^{2/3})^{-1}$. The RHS of Eqns. (33-34) can be reduced to double integrals using the same method of Eqn. (30), and are evaluated numerically. We can rearrange the sum in Eqn. (33) in the following form:

$$\sum_{r,s} T_{nrs}(m) = \sum_r a_r(m)\phi_r(m) = \sum_{r=1}^{\infty} A_r(m)\Delta^r\phi_n(m), \tag{36}$$

where Δ is the finite difference operator acting as follows:

$$\Delta^{2r+1}\phi_n = \frac{1}{2}(\Delta^{2r}\phi_{n+1} - \Delta^{2r}\phi_{n-1});$$

$$\Delta^{2r}\phi_n = \Delta^{2(r-1)}\phi_{n+1} + \Delta^{2(r-1)}\phi_{n-1} - 2\Delta^{2(r-1)}\phi_n, \tag{37}$$

so that the coefficients $A_r(m)$ are expressed in terms of the $a_r(m)$'s through the relation:

$$A_{2r+1} = a_{n+2r-1} - a_{n-2r+1}; \qquad A_{2r} = \frac{a_{n+2r-1} + a_{n-2r+1}}{2} \tag{38}$$

The fact that only differences between ϕ_n enter the expression above is due to the fact that we are expanding around a Kolmogorov spectrum, for which $T_n = 0$. For the same reason, the coefficients c_{nrs} and d_{nrs} are

invariant under the transformation: $\{n,r,s\}\to\{n+j,r+j,s+j\}$, which explains the fact that the finite difference coefficients A_r do not depend on the shell index n.

In Fig. 4 these coefficients are plotted in terms of the shell constant a. Notice that finite difference of order higher than 2 appear to be negligible for most choices of a. We obtain therefore the basic result that the shell dynamics obeys, for most values of a, a (discrete) heat equation forced by a random noise, with an advection term proportional to Δ.

The physical picture corresponding to this result is not new [29]. Fluctuations over finite volumes $\mathtt{B}_m$ moving with the flow, in the energy transfer to eddies at scale k_n, generate fluctuations in the energy content of these volumes and scales. The eddies are stretched by the turbulent flow, so that their energy (together with its fluctuation ϕ_n) is transferred towards smaller scales; the term responsible for this effect is the advection $A_1\Delta\phi_n$. At the same time, the randomness of the turbulent flow causes some eddies to be stretched more and some less, resulting in the end in a diffusion of energy over different scales.

IV. Anomalous scaling estimates

The fluctuations in the energy of wavepackets at different space locations is what is responsible for intermittency in this model. Instead of studying the scaling behaviors of the structure functions $S(l,q)=<v_l^q>$, we focus on the modified structure functions, defined through:

$$S(k_n^{-1},a,c_w,2q)=<<E_n>_n^q>=E_n^{(0)}<(1+\phi_n(n))^q>\,. \tag{39}$$

Identifying naively the space separation l in $S(l,q)$ with the wavelength k_n^{-1} in Eqn. (39), the two definitions of structure function coincide, for an appropriate choice of a, and for $c_w=c_w^{MAX}$. Given a large enough averaging volume V_i, the intermittency correction is small and we can linearize in n; in this way, using also: $k_n=k_0a^n$, we have:

$$\zeta_{2q}-q\zeta_2\simeq-\frac{1}{\ln a}\frac{\mathrm{d}}{\mathrm{d}n}<(1+\phi_n(n))^q>\,. \tag{40}$$

If we confine ourselves to the lowest order moments, large fluctuations of ϕ do not contribute too much, so that we can expand: $(1+\phi_n(n))^q\simeq1+\frac{q(q-1)}{2}<\phi_n(n)^2>$. Imposing the Kolmogorov relation for the third order structure function: $S(l,a,3)\propto l$, fixes the value for $\delta\zeta_2$, leading to the lognormal statistics result [6]:

$$\delta\zeta_q\simeq-\frac{q(q-3)}{4\ln a}\frac{\mathrm{d}<\phi_n(n)^2>}{\mathrm{d}n}; \tag{41}$$

we see then that the presence of anomalous scaling is associated with secular behavior of the fluctuations $\phi_n(n)$.

Unfortunately, the energy balance for the wavepackets, Eqn. (22) is not an equation for $\phi_n(n)$, but one for $\phi_n(m)$ for fixed m. We can rewrite Eqn. (22) in a more explicit form, by approximating the relaxation term $T_n(m)$ with the advection-diffusion operator introduced through Eqns. (35-37), and by rewriting the noise term f_n as a difference of energy flux fluctuations at different scales, as from Eqn. (25). After adequate rescaling, we are left with the equation:

$$[\exp(-\gamma n)\partial_{\hat t}-D\Delta^2+V\Delta)]\phi_n(m,t)=\hat\Delta(F_n(m)^{1/2}\xi), \tag{42}$$

where $\hat\Delta\xi_n=\xi_{n+1}-\xi_n$ and:

$$\hat t=t\,\bar\rho\bar\epsilon^{1/3}k_0^{2/3};\qquad\gamma=\frac{2}{3}\ln a;\qquad D=\frac{C_{Kol}}{\rho\bar\rho(1-a^{-2/3})}A_2;$$

$$V=\frac{C_{Kol}}{\rho\bar\rho(1-a^{-2/3})}A_1;\qquad F\simeq0.83\frac{(c_w(a-1))^3C_{Kol}\,a^{l(m,n)}}{(\bar\rho(1-a^{-2/3}))^2};$$

$$<\xi_{n+\frac{n'}{2}}(m,\hat t)\xi_{n-\frac{n'}{2}}(m,0)>\simeq\frac{\exp(-e^{\gamma n}|\hat t|)}{1+112.5(\gamma n')^2}. \tag{43}$$

Here the Kolmogorov and time scale constants C_{Kol} and ρ are the ones measured in the moving frame: $C_{Kol} \simeq 0.8$ and $\rho \simeq 0.3$ (it appears however that all final results do not change by more than 10% by exchanging these values with their Eulerian counterparts: $C_{Kol} \sim 1.5$ and $\rho \simeq 0.43$). The dependence of the noise correlation $< \xi\xi >$ on $n - n'$ in the formula above is a fit of the result of numerical integration shown in Fig. 3.

Clearly, Eqn. (42) does not lead to intermittent behaviors; the factor $a^{l(m,n)} = a^{(m-n)}$ in the noise amplitude, which goes to zero at small scales prevents it. Passing from Eqn. (42) to an equation for $\phi_n(n)$ requires the introduction of corrections due to the fact that now, wavepackets associated with different scales do not overlap exactly in real space. There are two such contributions:

$$(-D\Delta^2 + V\Delta)\phi_n(m)|_{m=n} - (-D\Delta^2 + V\Delta)\phi_n(n)$$
$$= (-D + V/2)(\phi_{n+1}(n+1) - \phi_{n+1}(n)) \qquad (44a)$$

and

$$\hat{\Delta}(F^{1/2}(n,m)\xi_n(m))|_{n=m} - \hat{\Delta}(F^{1/2}(n,n)\xi_n(n)) =$$
$$F^{1/2}(n,n)(\xi_{n+1}(n+1) - \xi_{n+1}(n)). \qquad (44b)$$

The shell equation for $\phi_n \equiv \phi_n(n,t)$ takes then the form (to simplify notations, the hat on the rescaled time $\hat{t}$ will be dropped in the following):

$$[\exp(-\gamma n)\partial_t - D\Delta^2 + V\Delta)]\phi_n = F_n^{1/2}(\hat{\Delta}\xi + \delta\xi), \qquad (45)$$

with $\delta\xi$, which is equal to the sum of the RHS's of Eqns. (43a-b), providing a new fluctuation source beside the original term $\hat{\Delta}\xi$.

At this point we are in the condition to identify the terms in the shell energy equation which are responsible for the generation of intermittency. For the sake of simplicity, and to make contact with the model of Eggers [22], let us adopt for a moment the approximation: $\xi_n \simeq e^{-\gamma n/2}\hat{\xi}$ with $< \hat{\xi}(t)\hat{\xi}(t') >= \delta(t-t')$, and similar equation for $\delta\xi$. The RHS of Eqn. (45) is then proportional to:

$$\hat{\Delta}\hat{\xi} - \gamma\hat{\xi}/2 + \delta\hat{\xi}. \qquad (46)$$

We see then that Eqn. (45), at statistical equilibrium, is very similar to a random walk equation in which the role of the time is played by n/V and that of the random kicking by $-\gamma\hat{\xi}/2 + \delta\hat{\xi}$. This noise term continuously pumps into the system fluctuations, which are dissipated at very large n, by viscosity. It is the random walk character of the process that leads to the linear growth of $< \phi_n^2 >$, with respect to n, already observed in [22]. The two terms providing the source of intermittency have different physical origin. The term $\delta\xi$, which comes from Eqns. (44a,b), is due to the competition in the energy transfer between eddies at different locations, characteristic of the Random Beta Model [8]. The term $\gamma\hat{\xi}/2$ instead, comes from the mismatch in the characteristic time scales of the energy transfer between different shells and was responsible for the production of intermittency in the stochastic model of Eggers [22-23].

The first term in Eqn. (46), which is the derivative of a random noise, produces the gaussian part of the fluctuations in the energy content of the wavepackets. Notice that this same quantity can be calculated directly from finite volume averages of $|\mathbf{v}^{(0)}|^2$, given the expression for the correlation given by Eqn. (3), together with Eqn. (18). This fact will be used to provide a check on the goodness of the approximations used to arrive to equation (45). Notice finally how in this approach, the smallness of the anomalous corrections is associated with the smallness of the parameter $a-1$ and with the fact that the amplitude of the intermittency source term is second order in this quantity, with respect to the source term of the gaussian fluctuations.

A. Solution of the shell energy equation

It is possible to solve Eqn. (45) either following [22], by diagonalizing the LHS of (45), considered as a matrix equation, or using the multiplier technique adopted in [23]. Here, the smallness of the parameter $a-1$ allows to consider the continuous limit of Eqn. (45) and to use a multiple scale expansion in which, to lowest order, the dependence on n produced by the $\exp(\gamma n)$ terms is neglected on the scale of the fluctuations. This

allows a solution of the problem in terms of green functions, in which no approximation on the form of the noise correlation is required. The green function for Eqn. (45) is:

$$g(n,n',t)=\frac{\exp\Big[-\frac{(n-n'-V\,e^{\gamma n}t)^2}{4D\,e^{\gamma n}t}+\frac{\gamma n}{2}\Big]}{\sqrt{4\pi D\,t}}, \tag{47}$$

The first quantity that we are going to calculate is the gaussian part of the fluctuations:

$$\begin{aligned}<\phi_G^2>&=F\int\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\omega}{2\pi}|g_{k\omega}(n)|^2k^2\ <|\xi_{k\omega}|^2>\\ &=\frac{F}{2D}\int\frac{\mathrm{d}k}{2\pi}\frac{(1+Dk^2)\exp(-\frac{0.14}{a-1}|k|)}{1+(V^2+2D)k^2+D^2k^4},\end{aligned} \tag{48}$$

where $g_{k\omega}(n)=(-i(\omega\,e^{-\gamma n}-Vk)+Dk^2)^{-1}$ is the Fourier transform with respect to n' and t of $g(n,n',t)$. As mentioned before, we can repeat the calculation directly from the statistics of the velocity field $\mathbf{v}^{(0)}$. Considering the limit of c_w small, corresponding to wavepackets thinner than the shell, nondiagonal contributions in Eqn. (18) can be neglected and we have:

$$\begin{aligned}<\phi_G^2>&=\frac{2}{E_n^{(0)2}}\int_n\frac{\mathrm{d}^3k}{(2\pi)^3}C_k^2\int_n\frac{\mathrm{d}^3q}{(2\pi)^3}|W_q(n)|^2\\ &\simeq 0.032\frac{(1-a^{-\frac{13}{3}})(c_w(a-1))^3}{(1-a^{-\frac{2}{3}})^2}.\end{aligned} \tag{49}$$

The two expressions for $<\phi_G^2>$ given by Eqns. (48) and (49) are plotted in Fig. 5. against a; the best agreement, though still rather rough, is obtained for the range $1.2\le a\le 1.3$, which is consistent with what would be expected by looking at the energy transfer profile (Fig. 2.) and at the one for the correlation of the energy flux fluctuations (Fig. 3.). In the same way it is possible to calculate the correlation time for ϕ_G:

$$\begin{aligned}\tau_c&=\frac{1}{<\phi_G^2>}\int_0^\infty\mathrm{d}t\ <\phi_G(0)\phi_G(t)>\\ &=\frac{-iF}{<\phi_G^2>}\int\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\frac{k^2}{\omega}|g_{k\omega}(n)|^2<|\xi_{k\omega}|^2>.\end{aligned} \tag{50}$$

The prediction from direct calculation, obtained from the generalization of Eqn. (49) to 2-time correlations, is $\tau_c(k_0)\simeq 1$, corresponding, in non rescaled units, to $\tau_c(k)^{-1}=2\eta_k$; here we find that $0.9\le\tau_c(k_0)\le 1$ in the whole range $1.1\le a\le 2$.

We turn next to the calculation of the intermittent part of the fluctuation ϕ. Care must be taken now due to the divergent nature of correlations at large n, which forbids in particular the use of the Fourier representation adopted in Eqn. (48).

The equation giving the growth of $<\phi_n^2>$ at large n is obtained by multiplying Eqn. (45) by ϕ and taking the average:

$$<\phi_n[\exp(-\gamma n)\partial_t-D\Delta^2+V\Delta]\phi_n>=F^{1/2}<\phi_n(\hat{\Delta}\xi_n+\delta\xi)>. \tag{51}$$

We subtract from Eqn. (51) the gaussian part of the fluctuations as given by Eqn. (48); then the finite difference $\hat{\Delta}$ acts only on the part of the noise variation due to the scaling of the correlation time, which is of order $a-1$. Next, we notice that to lowest order, the contribution to $\delta\xi$ coming from Eqn. (43a) is obtained by approximating ϕ by its gaussian component ϕ_G. The RHS of Eqn. (51) takes then the form:

$$\Xi(a,c_w)=F\int_0^\infty\mathrm{d}t\int\mathrm{d}n'\,[g(n,n',t)\,(\gamma^2(1-t^2)+\beta)$$

$$+g(n,n'-1,t)\beta' t] \ < \xi_n(0)\xi_{n+n'}(t) > +\beta < \phi_G(D-\frac{V}{2})\phi_G >, \tag{52}$$

where β and β' are $\mathcal{O}((a-1)^2)$ quantities giving respectively the amount of wavepacket volumes not overlapping (Beta model effect), and the correlation between this effect and that of the scale dependence of the noise correlation time. Explicit expressions for these coefficients and further simplification of Eqn. (52) are presented in Appendix C.

Next turn to the LHS of Eqn. (51). The time derivative term is equal to zero at steady state, while it is shown in Appendix C that the intermittent part of the fluctuations does not contribute to the $< \phi\Delta^2\phi >$ term. We are left then with: $< \phi\Delta\phi >\simeq \frac{1}{2}\Delta < \phi^2 >$, so that we obtain the result, for the kurtosis scaling exponent:

$$2\zeta_2-\zeta_4 \simeq \frac{1}{\ln a}\frac{\mathrm{d} < \phi^2 >}{\mathrm{d}n} = \frac{2\,\Xi(a,c_w)}{V(a)\,\ln a}. \tag{53}$$

The dependence of $2\zeta_2-\zeta_4$ on a for fixed c_w is shown in Fig. 6.; notice the saturation occurring at $a \simeq 1.3$, suggesting that the bulk of intermittency production occurs at scales of the order of three to ten times the size of the eddies in exam. From inspection of Eqns. (43), (52), (53) and (C3-6), we see that, for small $a-1$: $2\zeta_2-\zeta_4 = \mathcal{O}(c_w^3(a-1)^3)$, so that we obtain a direct connection between the smallness of the intermittency corrections and their being proportional to a rather large power of the small parameter $a-1$. Unfortunately, the intermittency exponents defined here depend sensitively on the parameter c_w; due to the difficulty in determining c_w^{MAX} with precisions, it is therefore problematic to make accurate predictions on the scaling of $S(l,q)$ starting from $S(l,a,c_w,q)$.

A value $2\zeta_2-\zeta_4 \simeq 0.016$, is obtained setting $c_w = 2$, corresponding to the "reasonable" condition that the noise granularity be equal to the shell thickness $(a-1)k_n$, i.e., from Eqns. (19) and (29): $W^2_{\mathbf{p}+\mathbf{q}-\mathbf{k}} \sim \exp(-\frac{|\mathbf{p}+\mathbf{q}-\mathbf{k}|^2}{2(k_{n+1}-k_n)^2})$. However choices as reasonable as the one just considered, lead to results differing from one another by up to an order of magnitude, so that the above formula should not be taken too seriously.

V. Summary and conclusions

We have described a model for the production of intermittency in the inertial range of three dimensional turbulence, based on statistical closure of the Navier Stokes equation. A connection between Navier Stokes dynamics and phenomenological models like the Random Beta Model [8], and the stochastic chains considered in [22-23], is in this way established. Although this connection may be rather tenuous, because of the assumptions adopted in deriving the closure, it is still pleasing that the results in this paper are obtained as lowest order corrections to a mean field approximation, which by itself would produce Kolmogorov scaling. It should also be mentioned that the present approach is able to produce dynamically, values of non-intermittent part of the energy fluctuations in agreement with the prediction from quasigaussian statistics of the velocity field; this is a bonus, which provides an indirect check on the goodness of the closure technique adopted.

The use of wavepackets rather than Fourier modes, is the reason why a perturbative treatment of intermittency has been possible here. There has been indeed an intriguing aspect in this subject, namely, the difficulty in associating, to the smallness of the anomalous corrections, a small parameter in which to carry on perturbation theory. One of the results of the present model is the identification of this parameter with the physical quantity $\Delta k/k$, i.e. the ratio between eddy size and the scale of the characteristic flow straining the eddy. Scaling corrections appear to be of third order in this quantity, with equally important contributions from Random Beta Model kind of effects [8], and from the mechanism of intermittency production of the model studied by Eggers [22-23].

The main result of this paper justifies a posteriori the use of wavepackets in the problem: the saturation in the a-dependence of the generalized structure functions $S(l,a,c_w,q)$, for $a \geq 1.3$, which is consistent with a separation of scales between production of intermittency and energy transfer. This is a definite prediction of the model, which, together with predictions on the actual magnitude of scaling corrections, could be tested by direct analysis of experimental data in terms of the generalized structure functions $S(l,a,c_w,q)$. This would extend similar studies, carried on by Menevau [30] using wavelet analysis.

We have not been able yet to establish a quantitatively accurate relation between the two structure function definitions: $S(l,a,c_w,q)$ and $S(l,q)$, the reason being the sensitive dependence of $S(l,a,c_w,q)$ on

$c_w = \frac{\Delta k}{(a-1)k}$, the ratio of wavepacket to shell thickness in k-space. Preliminary estimates hint towards a value for the scaling corrections smaller by a factor of the order of five than the experimental one. In the lognormal approximation: $\frac{\delta\zeta_q^{theo.}}{q(q-3)} \sim -.002$, which should be compared with the lognormal fit of experimental values: $\frac{\delta\zeta_q}{q(q-3)} \sim -.01$ [2]. Due to uncertainties in the present analysis, it is not possible yet to state whether inertial range processes are as important for intermittency production as finite size corrections, or if they are themselves just corrections to dominant finite size effects. We reiterate however that such an answer could be obtained analyzing experimental data in terms of $S(l, a, c_w, q)$ instead of $S(l, q)$.

Turning our attention to more formal issues, it is interesting to look for similarities between the model described in this paper and the various phenomenological approaches that have been used to study infinite Reynolds number intermittency. The basic ingredient here is the partition of Fourier space in shells of exponentially increasing radii. For this reason, our approach has a lot in common with the deterministic Shell Models studied, among others, by Yamada and Ohkitani [31] and by Jensen, Paladin and Vulpiani [32]. An interesting interpretation of these models, particularly clear in Zimin [33], comes from looking at the dynamical variable in each shell as the velocity of nested eddies, all located at the same space position, and interacting with one another locally in both scale and space. In our approach, such dramatic restriction of phase space does not take place, in that the behavior of the various wavepackets in a given shell is treated in an average sense; it is then possible that our model may underestimate the amount of intermittency produced in the inertial range. It remains to be seen how important this difference is; in principle one should compare the present model with some closure for the Yamada-Ohkitani model or, viceversa, look for a deterministic dynamical system, whose closure coincide with the one described in this paper, and then compare with the Yamada-Okhitani system.

The connection with the stochastic model of Eggers [22-23] is clearly simpler. In both cases one ends up with the same stochastic chain, the only difference being here the not exact conservation of energy transferred between shells. If this effect, due to the not exact overlapping of wavepackets at different scales, were neglected, the present model and the one of Eggers would coincide.

Going back to the possibility of turbulent Navier Stokes dynamics taking place in a very limited region of phase space, this constitutes the main conceptual limitation of our model. The possibility of preferential transfer of energy between individual wavepackets would correspond to intermittency being associated with locally anisotropic fluctuations. It should be mentioned that this looks somewhat unlikely, due to the separation of scales described earlier and the consequent large number of eddies in one straining region δk^{-1}. In any case this is a possibility, which could lead, in a suggestive way, to a mechanism for the creation of coherent structures. In [34], She derived anomalous exponents, based on assumptions on the shape of the most intense and rare dissipation regions, so that intermittency and coherent structures may be connected. It is difficult to see how shell models, of either the deterministic or the stochastic variety, could be used to study such coherent structures, given the very distant scales involved in their dynamics. In any case, perhaps this is not a big problen, since, although vortex tubes have been observed in many numerical simulations [3], they do not seem to contribute much to energy dissipation and their relevance to the turbulent dynamics has been questioned recently in [35].

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Appendix A. Quasi Lagrangean closure

The basic difficulty in dealing with Quasi Lagrangean closures is that studying the Navier Stokes dynamics in a reference frame moving with a single Lagrangean tracer, neglects the divergence of trajectories of different tracers. In a more refined closure scheme, this effect would be included by associating each point in a correlation function to a different tracer trajectory, like in the Lagrangean History DIA (LHDIA) of Kraichnan [25]. However, also in that theory, abridgements of the closure equations were necessary in the end, which were similar in their effect to neglecting the divergence of Lagrangean trajectories.

We try here to implement this approximation in as much a consistent way as possible. Notice first that correlations are defined starting from an initial case in which the initial point lies on the Lagrangean trajectory $\mathbf{z}_t$, so that Eqn. (3) basically describes the decorrelation of points at distance r from the Lagrangean trajectory $\mathbf{z}_t$, with respect to points lying on it. Since decorrelation occurs forward in time, this is the optimal choice; choosing $\mathbf{z}_t$ as the final point and $\mathbf{z}_0 + \mathbf{r}$ as the initial one, would lead in particular to no decorrelation, due to the advection term being zero in Eqn. (2).

The approximation of Eqn. (5) in which the initial point $\mathbf{z}_{t_1} + \mathbf{r}_1$ of the green function $G(t, \mathbf{r}|t_1, \mathbf{r}_1)$ is shifted on the Lagrangean trajectory, is justified with the previous choice. We try to make it more appealing by giving the next order in the expansion around $G(t, \mathbf{r}|t_1, \mathbf{r}_1) = G(\mathbf{r} - \mathbf{r}_1, t - t_1)$. We have first:

$$\begin{aligned}\int \mathrm{d}^3 r\, e^{-i\mathbf{kr}} < v^\alpha(\mathbf{r}_0, 0) v^\gamma(\mathbf{z}_t(\mathbf{r}_0, 0) + \mathbf{r}, t) > &= \int \mathrm{d}^3 r\, e^{-i\mathbf{kr}} \int \mathrm{d}^3 s\, G^{\gamma\rho}(0, \mathbf{r} + \mathbf{s}|t, \mathbf{r}) C^\alpha_\rho(\mathbf{r} - \mathbf{s}) \\ &= \int \mathrm{d}^3 r \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} G^{\gamma\rho}_{\mathbf{k}_1}(\mathbf{r}, t) C^\alpha_{\mathbf{k}_1,\rho}\, e^{i(\mathbf{k}_1 - \mathbf{k})\mathbf{r}}, \end{aligned} \tag{A1}$$

where $G^{\gamma\rho}_{\mathbf{k}_1}(\mathbf{r}, t)$ is the Fourier transform of $G^{\gamma\rho}(0, \mathbf{r} + \mathbf{s}|t, \mathbf{r})$ with respect to $\mathbf{s}$. Expanding the argument of the integral in the last line of Eqn. (A1) in $\mathbf{k} - \mathbf{k}_1$ and $\mathbf{r}$, we get then:

$$\begin{aligned}&\int \mathrm{d}^3 r\, e^{-i\mathbf{kr}} < v^\alpha(\mathbf{r}_0, 0) v^\gamma(\mathbf{z}_t(\mathbf{r}_0, 0) + \mathbf{r}, t) > \\ &= G^{\gamma\rho}_{\mathbf{k}}(0, t) C^\alpha_{\mathbf{k},\rho} - \frac{1}{2} \frac{\partial}{\partial k^\phi} \frac{\partial}{\partial k^\psi} \frac{\partial}{\partial r_\phi} \frac{\partial}{\partial r_\psi} \Big[G^{\gamma\rho}_{\mathbf{k}}(\mathbf{r}, t) C^\alpha_{\mathbf{k},\rho} \Big]_{\mathbf{r}=0} + ... \end{aligned} \tag{A2}$$

More in general we would obtain:

$$\begin{aligned}&\int \mathrm{d}^3 r\, e^{-i\mathbf{kr}} < v^\alpha(\mathbf{z}_{t_1}(\mathbf{r}_0, 0) + \mathbf{r}_1, t_1)) v^\gamma(\mathbf{z}_{t_1+t}(\mathbf{r}_0, 0) + \mathbf{r}_1 + \mathbf{r}, t_1 + t) > \\ &= G^{\gamma\rho}_{\mathbf{k}}(\mathbf{r}_1, t) C^\alpha_{\mathbf{k},\rho} - \frac{1}{2} \frac{\partial}{\partial k^\phi} \frac{\partial}{\partial k^\psi} \frac{\partial}{\partial r_\phi} \frac{\partial}{\partial r_\psi} \Big[G^{\gamma\rho}_{\mathbf{k}}(\mathbf{r}_1 + \mathbf{r}, t) C^\alpha_{\mathbf{k},\rho} \Big]_{\mathbf{r}=0} + ... \end{aligned} \tag{A3}$$

which can be then be expanded in Taylor series around $\mathbf{r}_1 = 0$. This shows that the lowest order approximation adopted in Eqn. (5) is rather bad and that our closure could not be expected to lead to quantitative accurate results.

We turn next to the derivation of the energy balance equation in the laboratory frame, Eqn. (10). We consider just one term:

$$\begin{aligned}&-\frac{1}{(2\pi)^3} \int \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \mathrm{d}^3 r_3\, e^{-i(\mathbf{kr}_1 - \mathbf{pr}_2 - \mathbf{qr}_3)} < v^{(0)\alpha}(\mathbf{r}_1, 0) v^{(0)}_\beta(\mathbf{r}_2, 0) \partial^\beta v^{(1)}_\alpha(\mathbf{r}_3, 0) > \\ &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \mathrm{d}^3 r_3\, e^{-i(\mathbf{kr}_1 - \mathbf{pr}_2 - \mathbf{qr}_3)} \int \mathrm{d}^3 s \int_{-\infty}^0 \mathrm{d}\tau\, G^\rho_\alpha(\mathbf{s}, -\tau) \\ &\times < v^\alpha(\mathbf{r}_1, 0) \mathbf{v}_\beta(\mathbf{r}_2, 0) \partial^\beta [v^\sigma(\mathbf{z}_\tau + \mathbf{r}_3 + \mathbf{s}, \tau) - \hat{w} v^\sigma(\mathbf{z}_\tau, \tau)] \partial_\sigma v_\rho(\mathbf{z}_\tau + \mathbf{r}_3 + \mathbf{s}, \tau) > . \end{aligned} \tag{A4}$$

Splitting the 4-point correlation into 2-point ones, we obtain, in terms of Fourier transforms:

$$2\delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \int \mathrm{d}^3 r_2 \mathrm{d}^3 r_3 \int \mathrm{d}^3 k_1 \mathrm{d}^3 k_2 \mathrm{d}^3 k_3 \delta(\mathbf{k} + \mathbf{p} + \mathbf{k}_2 - \mathbf{k}_3) \theta_{k_1 k_2 k_3} C_{k_2} C_{k_3}$$

$$\times\Big\{B_4(k_1k_2k_3)[\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{q}+2\mathbf{k}_2+\mathbf{k}_3)-w\delta(\mathbf{k}_1-\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{q}+\mathbf{k}_2+\mathbf{k}_3)]$$

$$+B_2(k_1k_2k_3)[\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\mathbf{k}+q+2\mathbf{k}_2+\mathbf{k}_3)-w\delta(\mathbf{k}_1-\mathbf{k}_2)\delta(\mathbf{k}+\mathbf{q}+2\mathbf{k}_2)]\Big\}, \tag{A5}$$

where:

$$B_2(k_1k_2k_3)=P^{\alpha\rho}(k_1)P_{\alpha\rho}(k_2)k_1^\beta P_{\beta\sigma}(k_3)k_2^\sigma=k_1k_2(1+z^2)(z+xy), \tag{A6}$$

and

$$B_4(k_1k_2k_3)=k_2^\sigma P_{\sigma\alpha}(k_1)P^{\alpha\rho}(k_2)P_{\rho\beta}(\mathbf{k}_3)k_1^\beta=-k_1k_2xz(x+yz). \tag{A7}$$

The various terms in Eqn. (A5) are tracked back to Eqn. (A4) as follows: the terms in B_4 come from the contraction $< v^\alpha(v^\sigma-wv^\sigma)>\partial^\beta G_\alpha^\rho<v_\beta\partial_\sigma v_\rho>$; those in B_2, from the remaining contraction; the terms in w come from the shift to Lagrangean frame. Notice now that, from Eqns. (A6-7): $B_2(k_1k_1k_3)=B_4(k_1k_2k_1)=0$. Thus the terms in w in Eqn. (A5) disappear and one is left with the same expression that would be obtained from Eulerian DIA [24], but with the Lagrangean response time θ. Repeating the same calculation with the other two choices for $v^{(1)}$, we see that no terms in w contribute and we obtain the standard result of Eqn. (10).

We pass to the calculation of the 2-time Lagrangean correlation: at steady state the part of the time integrals from $\tau<0$ do not contribute and we have:

$$-\int \mathrm{d}^3r\, e^{-i\mathbf{kr}}<v^\alpha(\mathbf{r}_0,0)[v^\beta(\mathbf{z}_t+\mathbf{r},t)-\hat{w}v^\beta(\mathbf{z}_t,t)]\partial_\beta v_\alpha(\mathbf{z}_t+\mathbf{r},t)>$$

$$\begin{aligned}=\int \mathrm{d}^3re^{-i\mathbf{kr}}\int_0^t\mathrm{d}\tau\int\mathrm{d}^3s\Big\{&G_\rho^\beta(\mathbf{s},t-\tau)\Big[[C^{\alpha\sigma}(-\mathbf{r}-\mathbf{s},\tau)-\hat{w}C^{\alpha\sigma}(0,\tau)]\partial_\sigma\partial_\beta C_\alpha^\rho(-\mathbf{s},t-\tau)\\
&+\partial_\beta[C_\alpha^\sigma(-\mathbf{s},t-\tau)-\hat{w}C_\alpha^\sigma(\mathbf{r},t-\tau)]\partial_\sigma C^{\rho\alpha}(-\mathbf{r}-\mathbf{s},\tau)\Big]\\
&-\hat{w}G_\rho^\beta(\mathbf{s},t-\tau)\Big[[C^{\alpha\sigma}(-\mathbf{s},\tau)-\hat{w}C^{\alpha\sigma}(0,\tau)]\partial_\sigma\partial_\beta C_\alpha^\rho(\mathbf{r}-\mathbf{s},t-\tau)\\
&+\partial_\beta[C_\alpha^\sigma(\mathbf{r}-\mathbf{s},t-\tau)-\hat{w}C_\alpha^\sigma(\mathbf{r},t-\tau)]\partial_\sigma C^{\rho\alpha}(-\mathbf{s},\tau)\Big]\\
&+\partial_\beta G_{\alpha\rho}\Big[[C^{\alpha\sigma}(-\mathbf{r}-\mathbf{s},\tau)-\hat{w}C^{\alpha\sigma}(0,\tau)]\\
&\times\partial_\sigma[C_\rho^\beta(-\mathbf{s},t-\tau)-\hat{C}_\rho^\beta(-\mathbf{r}-\mathbf{s},t-\tau)]\\
&+\partial_\sigma C^{\alpha\rho}(-\mathbf{r}-\mathbf{s},\tau)[C^{\beta\sigma}(-\mathbf{s},t-\tau)-\hat{w}C^{\beta\sigma}(\mathbf{r},t-\tau)\\
&-\hat{w}C^{\beta\sigma}(-\mathbf{r}-\mathbf{s},t-\tau)+\hat{w}\hat{w}C^{\beta\sigma}(0,t-\tau)]\Big]\Big\}.\end{aligned} \tag{A8}$$

Of the terms on the RHS of Eqn. (A8), there is a group which does not depend on the integration variable $\mathbf{s}$; these terms come from integrating along the Lagrangean trajectory and lead, after Fourier transform, to triads in which one of the wavevectors is zero; as before they do not contribute to the final result. The remaining terms in $\hat{w}$ come from the $\hat{w}$ on the LHS of the equation, which express the fact that we are dealing with a Lagrangean correlation. These terms remain and are responsible for the cancellation of the sweep part of the correlation decay. After Fourier transorm, we obtain the result:

$$\int_0^t\mathrm{d}\tau\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})$$

$$\begin{aligned}&\times\Big[B_3(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau)-w_pB_3(q,-p,k)G_p(t-\tau)C_k(t-\tau)C_q(\tau)]\\
&+[B_1(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau)-w_pB_1(q,-p,k)G_p(t-\tau)C_k(t-\tau)C_q(\tau)]\\
&+[B_4(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau)-w_qB_4(p,k,-q)G_k(t-\tau)C_p(\tau)C_q(t-\tau)]\end{aligned}$$

$$+B_2(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau) - w_q B_2(p,k,-q)G_k(t-\tau)C_p(\tau)C_q(t-\tau)]\Big], \tag{A9}$$

where:

$$B_1(kpq) = p_\sigma P^{\sigma\alpha}(q)P_{\alpha\rho}(k)P^{\rho\beta}(p)k_\beta = -kp\, yz(y+xz) \tag{A10}$$

and

$$B_3(kpq) = k_\beta P^{\beta\rho}(p)P_{\rho\alpha}(q)P^{\alpha\sigma}(k)p_\sigma = kp\, xy(1-z^2). \tag{A11}$$

(The same notation of Leslie [26] is used here for the functions B_i, $i = 1, ...4$). The terms on the RHS of Eqn. (A9) are ordered as on the RHS of Eqn. (A8), once the terms of the last one, that are equal to zero, are eliminated. Using the following relations:

$$B_3(q,-p,k) = -B_1(kpq); \qquad B_1(q,-p,k) = -B_3(kpq)$$

$$B_4(p,k,-q) = B_1(kpq); \qquad B_2(p,k,-q) = B_2(kpq), \tag{A12}$$

and the definitions:

$$b_{kpq} = \frac{1}{2k^2}\sum_i B_i(kpq); \qquad b^{(1)}_{kpq} = \frac{1}{2k^2}(B_1(kpq) + B_3(kpq));$$

$$b^{(2)}_{kpq} = \frac{1}{2k^2}(B_1(kpq) + B_2(kpq)), \tag{A12}$$

and substituting into Eqn. (A9), we obtain immediately the result of Eqn. (11).

Finally, we derive the energy equation in the moving reference frame. Using conservation of energy triad by triad (which is preserved, together with incompressibility, when passing to the Lagrangean reference frame), we can write:

$$D_t C_k(t)|_{t=0} = \frac{1}{4\pi}\int_\Delta \mathrm{d}p\mathrm{d}q\, kpq\theta_{kpq}[-H(kpq) + H(pkq) + H(qkp)], \tag{A13}$$

where:

$$H(kpq) = [b_{kpq} + w_p b^{(1)}_{kpq}]C_k C_q - w_q b^{(2)}_{kpq} C_p C_q. \tag{A14}$$

We obtain immediately:

$$D_t C_k(t)|_{t=0} = \frac{1}{4\pi}\int_\Delta \mathrm{d}p\mathrm{d}q\, kpq\theta_{kpq}[(b_{kpq} + w_q b^{(3)}_{kpq})C_p C_q$$

$$-(b_{kpq} + (\frac{p}{k})^2 w_q b^{(3)}_{p,k,-q})C_k C_q], \tag{A14}$$

where:

$$b^{(3)}_{kpq} = \frac{1}{2k^2}(2B_2(kpq) + B_1(kpq) + B_4(kpq)). \tag{A15}$$

A form of Eqn. (A14), which is manifestly zero at equipartition, can be obtained using the relations:

$$B_2(kpq) = B_2(p,k,-q) \qquad \text{and} \qquad B_1(kpq) = B_4(p,k,-q). \tag{A16}$$

Substituting Eqns. (A15-16) into Eqn. (A14), we obtain finally the result shown in Eqn. (15).

Appendix B. Noise correlation

In order to calculate the amplitude of the energy flux fluctuation $g_n(m)$, we must evaluate the contractions of the product:

$$v^\alpha(\mathbf{z}_1)[v^\beta(\mathbf{z}_1+\mathbf{r}_1) - v^\beta(\mathbf{z}_1)]\partial_\beta v^\alpha(\mathbf{z}_1+\mathbf{r}_1)v^\gamma(\mathbf{z}_2)[v^\sigma(\mathbf{z}_2+\mathbf{r}_2) - v^\sigma(\mathbf{z}_2)]\partial_\sigma v^\gamma(\mathbf{z}_2+\mathbf{r}_2), \tag{B1}$$

which are listed in Table B1. In this section, we shall adopt the notation: $\psi_\alpha \equiv \partial_\alpha \psi$. Of these contractions, numbers (B1.1), (B1.4), (B1.7), (B1.10) and (B1.13) do not contribute because of incompressibility. The remaining ones fall into two groups: (B1.2), (B1.3), (B1.4) and (B1.6) are in the form: $< g(0)g(t) > \to < v(0)v(0) >< v(0)v(t) >< v(t)v(t) >$, where $< v(0)v(t) >$ is associated with velocity fluctuations on the scale of the wavepacket radius R. These terms come from noise components of g in the form $v < (v\nabla)v >$, which are identicaly zero, and $(v\nabla) < v^2/2 >$, which are associated with sweep and do not contribute to energy transfer. The second term contains all the other terms, which are in the form $< v(0)v(t) >< v(0)v(t) >< v(0)v(t) >$ and act on the same scale of k_1 and k_2.

It is possible to show explicitly that the contractions in the first group do not contribute to $< g^2 >$. Consider for example contraction (B1.3):

$$
\begin{aligned}
&< g_{n_1}(m_1,t_1) g_{n_2}(m_2,t_2) >^{(3)} = \int \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 H(r_1,n_1) H(r_2,n_2) \\
&\times \int \mathrm{d}^3 z_1 \mathrm{d}^3 z_2 W(z_1,m_1) W(z_2,m_2) [C^{\alpha\beta}(\mathbf{r}_1,0) - \hat{w} C^{\alpha\beta}(0,0)] \\
&\times [C^{\alpha\sigma}_{\beta}(\mathbf{z}_1 + \mathbf{r}_1 - \mathbf{z}_2 - \mathbf{r}_2, t) - \hat{w} C^{\alpha\sigma}_{\beta}(\mathbf{z}_1 + \mathbf{r}_1 - \mathbf{z}_2, t)] C^{\gamma\gamma}_{\sigma}(\mathbf{r}_2,0) \\
&= 2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} C_k C_p C_q \exp(-\eta_p |t|) W_{\mathbf{p}}(m_1) W_{\mathbf{p}}(m_2) \\
&\times \Big\{ \frac{(\mathbf{kp})^2 (\mathbf{pq})}{k^2 p^2} - \frac{(\mathbf{kp})(\mathbf{kq})}{k^2} \Big\} \\
&\times [(2 - w_k(k+p)) H(k + p - k_{n_1}) - w_k(k+p) H(p - k_{n_1})] \\
&\times [(2 - w_p(p+q)) H(p + q - k_{n_2}) - w_p(k) H(q - k_{n_2})].
\end{aligned} \tag{B2}
$$

The geometric term in braces is antisimmetric in $\mathbf{q}$, so that that only nonzero contributions are proportional to $H(p + q - k_{n_2})$. We change first variables: $q + p \to q$, so that $H(p + q - k_{n_2}) \to H(q - k_{n_2})$. Because of the factor $W_{\mathbf{p}}(m_1) W_{\mathbf{p}}(m_2)$, we can expand in a power series in p:

$$
-k_2^\alpha \partial_{k_3^\alpha} \Big[(kq)^{-5/3} \Big(\frac{(\mathbf{kp})^2 (\mathbf{pq})}{k^2 p^2} - \frac{(\mathbf{kp})(\mathbf{kq})}{k^2} \Big) \Big] = -\frac{5}{3} (kq)^{-5/3} p^2 (xyz - x^2 z^2) \tag{B3}
$$

Indicating by u the cosine of the angle between the planes $\mathbf{kp}$ and $\mathbf{kq}$, we have:

$$
xyz - x^2 z^2 = uxz \sqrt{(1 - x^2)(1 - z^2)},
$$

so that, substituting Eqn. (B3) into (B2) gives zero. Going to next order produces a factor $p^2 x^2$, which gives zero again, while still the next order leads to a contribution which is by a factor $(kR)^{2/3}$ smaller than the terms in the second group.

The calculation of the terms in the secon group is very tedious but straightforward and follows the same lines of Eqns. (28) and (B2). We conclude by giving the exact form of the coefficients entering the expression for the fluctuation amplitude $< g^2 >$ [Eqn. (30)]:

$$
\begin{aligned}
B_1(kpq) = -pk\, yz\, (y + xz); \qquad B_2(kpq) = kp\, (1 + z^2)(z + xy); \\
B_3(kpq) = kp\, xy\, (1 - z^2).
\end{aligned} \tag{B4}
$$

$$
\begin{aligned}
H_1 = &[(2 - w_p(k)) H(k - k_{n_1}) - H(q - k_{n_1}) w_p(k)] \\
&\times [[(2 - w_q(k)) H(k - k_{n_2}) - H(p - k_{n_2}) w_q(k)] \\
&-[(2 - w_q(p)) H(p - k_{n_2}) - H(k - k_{n_2}) w_q(p)]];
\end{aligned} \tag{B5}
$$

$$
\begin{aligned}
H_2 = &[(2 - w_p(k)) H(k - k_{n_1}) - H(q - k_{n_1}) w_p(k)] \\
&\times [[(2 - w_p(k)) H(k - k_{n_2}) - H(q - k_{n_2}) w_p(k)] \\
&-[(2 - w_p(q)) H(q - k_{n_2}) - H(k - k_{n_2}) w_p(q)]];
\end{aligned} \tag{B6}
$$

$$
\begin{aligned}
H_3 = &[(2 - w_p(k)) H(k - k_{n_1}) - H(q - k_{n_1}) w_p(k)] \\
&\times [[(2 - w_k(q)) H(q - k_{n_2}) - H(p - k_{n_2}) w_k(q)] \\
&-[(2 - w_k(p)) H(p - k_{n_2}) - H(q - k_{n_2}) w_k(p)]].
\end{aligned} \tag{B7}
$$

Appendix C. Coefficients for the shell energy equation

The RHS of Eqn. (52) contains the following contributions:
A term giving the effect of the timescale becoming shorter at large n:

$$-F\int_0^\infty \mathrm{d}t \int \mathrm{d}n' g(n,n',t)\, F(n')\partial^2_{n'}\Xi(n',t)$$

$$= F\int_0^\infty \mathrm{d}t \int \mathrm{d}n' g(n,n',t)(1-t^2) < \xi_n(0)\xi_{n+n'}(t) > . \tag{C1}$$

where:

$$\Xi(n',t) = \frac{< \xi_n(0)\xi_{n+n'}(t) >}{< \xi_n(0)\xi_{n+n'}(0) >}, \quad \text{and} \quad F(n') =< \xi_n(0)\xi_{n+n'}(0) > \tag{C2}$$

A term coming purely from the noise, due to Beta Model effect:

$$F\int_0^\infty \mathrm{d}t \int \mathrm{d}n' g(n,n'-n,t) < (\xi_n(n,0)-\xi_n(n-1,0))(\xi_{n'}(n',t)-\xi_{n'}(n'-1,t)) >$$

$$= F\beta\int_0^\infty \mathrm{d}t \int \mathrm{d}n' g(n,n',t) < \xi_n(0)\xi_{n+n'}(t) >; \tag{C3}$$

where:

$$\beta = 1 + a^{-3} - 2^{\frac{5}{2}}(1+a^2)^{-\frac{3}{2}}. \tag{C4}$$

A cross correlation term between Beta Model effect and timescale mismatch:

$$2F\int_0^\infty \mathrm{d}t \int \mathrm{d}n' g(n,n',t)\partial'_n\Xi(n',t) < [\xi_n(n,0)-\xi_n(n-1,0)]\xi_{n+n'-1}(0) >$$

$$= F\beta'\int_0^\infty \mathrm{d}t \int \mathrm{d}n' g(n,n',t) < \xi_n(0)\xi_{n+n'-1}(t) >, \tag{C5}$$

where:

$$\beta' = 2\gamma\Big(1 - \Big(\frac{1+a^{-2})}{2}\Big)^{\frac{3}{2}}\Big). \tag{C6}$$

Finally there is the correction to the relaxation terms of the shell equation, from Beta Model effect:

$$(D - \frac{V}{2}) < (\phi_n(n,0)-\phi_n(n-1,0))(\phi_n(n,t)-\phi_n(n-1,t)) >$$

$$\simeq \beta(D-\frac{V}{2}) < \phi_G^2 > . \tag{C7}$$

To lowest order, there are no cross correlation between $(D-\frac{V}{2})(\phi_G(n)-\phi_G(n+1))$ and previous terms due to the origin of ϕ_G from $\partial_{n'}\xi$ rather than from ξ.

Using the formula $\int_0^\infty \mathrm{d}t\ \exp(-\frac{b}{t^2}-ct^2) = \sqrt{\frac{\pi}{4c}}\exp(-2\sqrt{bc})$ [36], and shifting the n' integration so that $n \to 0$, the integral term in Eqn. (52) can be rewritten in the form:

$$\int \mathrm{d}n' \Big[< \xi_0(0)\xi_{n'}(0) > \Big[\gamma^2\Big(\frac{\partial^2}{\partial c^2}-1\Big)+\beta\Big]$$

$$+\beta' < \xi_0(0)\xi_{n'-1}(0) > \frac{\partial}{\partial c}\Big]\sqrt{\frac{\pi a'^2}{4c^2}}\exp(d-2\sqrt{bc}), \tag{C7}$$

where:

$$a' = \frac{1}{\sqrt{\pi D}}; \qquad b = \frac{n'^2}{4D};$$

$$c = \frac{V}{4D} + \exp(\gamma n'); \qquad d = \frac{n'V}{2D}. \tag{C8}$$

Writing explicitly:

$$\int \mathrm{d}n' \Big\{ \Big[\gamma^2 \Big(1 - \frac{32D^2}{(V^2+4D)^2} - \frac{6D|n'|}{(V^2+4D)^{\frac{3}{2}}} - \frac{n'^2}{V^2+4D} \Big) + \beta \Big] \frac{1}{1+50(a-1)^2 n'^2}$$
$$+\beta' \frac{4D + |n'|\sqrt{4D+V^2}}{4D+V^2} \frac{1}{1+50(a-1)^2(n'-1)^2} \Big\}$$
$$\times \frac{F}{\sqrt{4D+V^2}} \exp\Big[\frac{1}{2D}(n' - |n'|\sqrt{4D+V^2}) \Big]. \tag{C9}$$

It remains to evaluate the term $D < \phi\Delta^2\phi >$ on the LHS of Eqn. (51):

$$< \phi\Delta^2\phi >= \int_0^\infty \mathrm{d}t_1 \mathrm{d}t_2 \int \mathrm{d}n_1 \mathrm{d}n_2 \, F(n_1-n_2) g(n_1,t_1) \partial_{n_2}^2 g(n_2,t_2) \, \Xi[t_1-t_2]$$
$$= \sum_{m=0}^\infty \frac{\Xi_m}{m!} \int_0^\infty \mathrm{d}t \int \mathrm{d}n_1 \mathrm{d}n_2 \, F(n_1-n_2) g(n_1,t_1) \partial_{t_2}^m \partial_{n_2}^2 g(n_2,t_2), \tag{C10}$$

where

$$\Xi_m \simeq \int \mathrm{d}t \, t^m \Xi(t). \tag{C11}$$

Using the defining relation for g: $(\partial_t - D\partial_n^2 + V\partial_n) g(n,t) = \delta(n)\delta(t)$ (in the continuum limit) and the condition: $g(n,0) = g(n,\infty) = 0$, we can write:

$$< \phi\Delta^2\phi >= \sum_{m=0}^\infty \frac{\Xi_m}{m!} \int_0^\infty \mathrm{d}t \int \mathrm{d}n_1 \mathrm{d}n_2 \, F(n_1-n_2) g(n_1,t_1) (D\partial_{n_2}^2 - V\partial_{n_2})^m \partial_{n_2}^2 g(n_2,t_2)$$
$$= \sum_{m=0}^\infty \frac{\Xi_m}{m!} \int \mathrm{d}n_1 \mathrm{d}n_2 \, F(n_1-n_2) (D\partial_{n_2}^2 - V\partial_{n_2})^m \frac{n_1^2-n_2^2}{2D(n_1^2+n_2^2)^2}. \tag{C12}$$

Integrating by part we obtain a series in the form:

$$< \phi\Delta^2\phi >= \sum_{m=0}^\infty \frac{\Xi_m}{m!} \int \mathrm{d}n_1 \mathrm{d}n_2 \, \frac{n_1^2-n_2^2}{2D(n_1^2+n_2^2)^2} (a_m \partial_{n_2}^{2m} + b_m \partial_{n_2}^{2m-1}) F(n_1-n_2). \tag{C13}$$

Given the form of the correlation F [see Eqn. (43)], the action on it of the operators $(a_m \partial_{n_2}^{2m} + b_m \partial_{n_2}^{2m-1})$ will lead to terms in the form:

$$\frac{c_1 + c_2(n_1-n_2)}{(1+50(a-1)^2(n_1-n_2)^2)^p}. \tag{C14}$$

Substituting back into Eqn. (C13), we see that terms in c_1 do not contribute because of the antisimmetry of the integrand under the transformation $n_1 \to n_2$, $n_2 \to n_1$. Similarly for the terms in c_2, due to the antisimmetry under simultaneous change of sign of n_1 and n_2. Hence the term $< \phi\Delta^2\phi >$ does not contribute to Eqn. (51).

FIGURE CAPTIONS

Fig. 1. Kolmogorov constant C_{Kol} and dimensionless parameter ρ, as measured from the energy flux through scale k, in a reference frame moving with velocity $\mathbf{v}(\lambda', k) = \hat{w}(\lambda', k)\mathbf{v}$, $0 < \lambda' < 0.9$, with $\lambda = 0.9$, fixed [see Eqn. (14)].

Fig. 2. Energy transfer profiles $T(k,p)$ for $k = 1$. (a): Lagrangean frame using Eqn. (15); $\lambda' = 0.9$. (b): laboratory reference frame, using Eqn. (10). (c): Lagrangean frame using simplified closure with Eqn. (17) and: $w_{p,q} = w_{p,q}(0.7, k)$. ($d$): the same as ($c$), but with $w_p = w_p(0.7, \min(k,q))$ and $w_q = w_p(0.7, \min(k,p))$.

Fig. 3. Normalized noise correlation $\frac{<g_n g_{n+n'}>}{<g_n^2>}$ vs. shell separation $x = (a-1)n'$. (a): calculation from Eqn. (30); (b): fit of (a) by: $F(x) = (1 + 50x^2)^{-1}$. (c): profile for $A(x) = \int \mathrm{d}p |T(k,p)T(k+x, p+x)|$, normalized to $A(0)$;

Fig. 4. Finite difference coefficients A_r vs. a [see Eqns. (33), (36) and (37)]. Notice that higher order differences are always dominated by the advection-diffusion part.

Fig. 5. Comparison of gaussian part of the fluctuation amplitude $<< v_l^2 >_R^2> - << v_l^2 >^2>$, obtained from integration of Eqn. (49) (a), and from direct average of $v^{(0)}$ (b).

Fig. 6. Scaling correction to $S(l, a, c_w, 4)$, as a function of a for $c_w = 2$; notice the saturation occurring at $a \simeq 1.3$.

Table B1. Contractions contributing to $< g_n(m) g_{n'}(m') >$.